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Projective Geometry, Lagrangian Subspaces, and Twistor Theory

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Abstract

The use of projective geometry for the characterization of Lagrangian subspaces and maps among them is of particular interest for the symplectic manifold that is twistor space. We raise some conjectures on how these should be interpreted on the space-time manifold by making use of the structure of projective twistor space.

1. *Introduction*

A previous paper (Campbell and Dodson, 1979) discussed the use of projective geometry for characterizing the Lagrangian subspaces defined by real polarizations of $\mathbb{R}^{2n}$ and, in addition, maps among them. The central results contained there will be applicable to a symplectic manifold which is $\mathbb{C}^{2n}$ since these results stem from theorems of projective geometry which are valid for general vector spaces.

In particular, it is interesting to consider the implications of projective geometrical concepts for polarizations of twistor space (Penrose, 1975) which is $\mathbb{C}^4$. With the convention (Morrow and Kodaira, 1971) that $T(M)$ is the canonically defined *holomorphic tangent bundle* of a complex manifold M, we have the following

Definition 1. A polarization, F, of a smooth $2n$-dimensional symplectic complex manifold (M, ω) is a smooth distribution

$$F : M \to T(M) : m \mapsto F_m$$

such that, $\forall m \in M$,

(i) F_m is an n-dimensional subspace of the complex vector space T_mM with

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the property that it is maximally isotropic,

$$F_m \times F_m \subseteq \ker \omega$$

F_m is then a *Lagrangian subspace of* $T_m M$.

(ii) F is involutive; its vector fields form a Lie subalgebra. We can similarly define the conjugate polarization $\bar{F}$ to be a smooth distribution

$$\bar{F} : M \to \overline{T(M)}$$

where now $\overline{T(M)}$ is the *conjugate tangent bundle* of M.

Remarks

(i) Simms and Woodhouse (1976) describe the use of polarizations derived from the *complexification* of bundles over real symplectic manifolds in geometric quantization.

(ii) Tarski (1976) has claimed that elementary particle resonance states for higher spin are suitably defined by certain polarizations of the underlying phase space.

2. *Projective Geometry and Polarizations of Twistor Space*

For our purposes it is more natural to use the above definition of polarization. We take as our symplectic manifold $(\mathbb{C}^{2n}, \omega)$, where ω is, for example, the standard symplectic structure on $\mathbb{C}^{2n}$. Then, the subspaces $F_m \subset T_m\mathbb{C}^{2n}$ are essentially $(n-1)$-dimensional hyperplanes in the projective space $\mathbb{C}P^{2n-1}$ of $T_m\mathbb{C}^{2n}$. Indeed, the maximal isotropy property of these subspaces is characterized in the projective space by the action of the corresponding *symplectic correlation* $\mathscr{C}$. If ω is any symplectic structure on $\mathbb{C}^{2n}$ then by Theorems 1, 2 (Campbell and Dodson, 1976) it defines a symplectic correlation $\mathscr{C}$ on $\mathbb{C}P^{2n-1}$ by

$$\mathscr{C}(\mathbb{P}W) = \{\mathbf{v} \mid \omega(v, w) = 0, \ \forall w \in W\}$$

for proper subspaces $W \subset \mathbb{C}^{2n}$. [We conform to the notation of Penrose (1975) by adopting the prefix $\mathbb{P}$ for a projective space and a symbol like $\mathbf{v}$ for its projective points.]

> *Proposition* 1. A subspace $F_m \subset T_m\mathbb{C}^{2n}$ is maximally isotropic if and only if $\mathbb{P}F_m \subseteq \mathscr{C}(\mathbb{P}F_m)$.

Proof. By definition, an n-dimensional subspace $F_m \subset T_m\mathbb{C}^{2n}$ is maximally isotropic if $\omega(v, w) = 0$, $\forall v, w \in F_m$. Thus

$$\mathbb{P}F_m \subseteq \mathscr{C}(\mathbb{P}F_m)$$

Also

$$\mathbb{P}F_m \subseteq \mathscr{C}(\mathbb{P}F_m) = \{\mathbf{v} \mid \omega(v, w) = 0, \quad \forall w \in F_m\} \Rightarrow \omega(v, w) = 0, \quad \forall v, w \in F_m$$

so F_m is maximally isotropic. □

So those Lagrangian subspaces determined by a polarization F of a symplectic manifold $(\mathbb{C}^{2n}, \omega)$ can now be realized projectively as $(n-1)$-dimensional hyperplanes $\mathbb{P}F_m$ in each projective space $\mathbb{P}T_m\mathbb{C}^{2n}$. They have the defining property $\mathbb{P}F_m \subseteq \mathscr{C}(\mathbb{P}F_m)$, where $\mathscr{C}$ is the symplectic correlation on $\mathbb{P}T_m\mathbb{C}^{2n}$ defined by the symplectic structure on $\mathbb{C}^{2n}$.

Example. Flat twistor space, $\mathbb{T}$, is $\mathbb{C}^4$ with symplectic structure $idZ^\alpha \wedge d\bar{Z}_\alpha$, $\alpha = 0, \ldots, 3$; Z^α being a twistor, an object that is decomposable into spinor parts describing the momentum and angular momentum structure of a zero-rest-mass particle. Projective twistor space, $\mathbb{PT}$, is just $\mathbb{C}P^3$ and is constructed from points $\mathbf{Z}$ which are the equivalence classes of twistors obtained by multiplying given nonzero twistors Z^α by all nonzero complex numbers.

By the above, a polarization F of $\mathbb{T}$ assigns to each $Z^\alpha \in \mathbb{T}$ a complex projective line $\mathbb{P}F_{Z^\alpha}$ in the projective space $\mathbb{P}T_{Z^\alpha}\mathbb{T}$. Each such line has the property $\mathbb{P}F_{Z^\alpha} \subseteq \mathscr{C}(\mathbb{P}F_{Z^\alpha})$, where the symplectic correlation $\mathscr{C}$ on $\mathbb{P}T_{Z^\alpha}\mathbb{T}$ is defined by the symplectic structure of $\mathbb{T}$. More precisely, we have the following.

Lemma.

$$\mathbb{P}F_{Z^\alpha} = \mathscr{C}(\mathbb{P}F_{Z^\alpha})$$

Proof. This follows from the dimension theorem for a correlation of a projective space (Yale, 1968, p. 258), since by this theorem $\mathscr{C}(\mathbb{P}F_{Z^\alpha})$ will also be a complex projective line. □

3. *Projective Twistor Space and Minkowski Space-Time*

The projective ideas of Section 2 are particularly appropriate in twistor theory. To see this we must make use of the structure of $\mathbb{PT}$ and its relationship with compactified Minkowski space-time $\mathbb{M}$. Penrose (1967) has described this structure. In this section we shall summarize the main ideas after first giving some precise definitions, which make use of some previous results (Campbell and Dodson, 1976; Yale, 1968) concerning correlations of a projective space.

Definition 2. The Hermitian form on $\mathbb{T}$ is a map

$$h : \mathbb{T} \times \mathbb{T} \to \mathbb{C} : (X^\alpha, R^\alpha) \to X^\alpha \bar{R}_\alpha$$

where

$$X^\alpha \bar{R}_\alpha = X^0 \overline{R^2} + X^1 \overline{R^3} + X^2 \overline{R^0} + X^3 \overline{R^1}$$

By Theorem 1 (Campbell and Dodson, 1976) we can use this definition to state the following:

Definition 3. The *Hermitian correlation* $\mathscr{H}$ on $\mathbb{PT}$ is an inclusion reversing permutation of the proper subspaces of $\mathbb{PT}$ defined by

$$\mathscr{H}(\mathbb{P}W) = \{\mathbf{Z} \mid h(Z^\alpha, R^\alpha) = 0, \ \forall R^\alpha \in W\}$$

for proper subspaces $W \subset \mathbb{T}$.

Now any correlation of a projective space interchanges points with hyperplanes (Yale, 1968, p. 258). The hyperplane $\bar{\mathbf{Z}} \subset \mathbb{PT}$ that is "*conjugate*" to the point $\mathbf{Z}$ in this way is just

$$\bar{\mathbf{Z}} = \{\mathbf{X} \in \mathbb{PT} \mid h(X^\alpha, R^\alpha) = 0,\ \forall R^\alpha = \lambda Z^\alpha; \lambda \in \mathbb{C} - \{0\}\}$$

We can now state the following.

Definition 4.

$$\mathbb{PN} \equiv \{\mathbf{Z} \in \mathbb{PT} \mid \mathbf{Z} \text{ lies on } \bar{\mathbf{Z}}\}$$

Remarks

(i) It follows from the previous equation for $\bar{\mathbf{Z}}$ that $\mathbf{Z}$ lies on $\bar{\mathbf{Z}}$ if and only if $h(Z^\alpha, Z^\alpha) = Z^\alpha \bar{Z}_\alpha = 0$. So, $\mathbb{N} \subset \mathbb{T}$ is the space of *null twistors.*

(ii) Also, from the definition of a symplectic correlation (Yale, 1968, p. 266) the Hermitian correlation $\mathscr{H}$ is symplectic on $\mathbb{PN}$.

(iii) $\mathbb{PT}$ can be decomposed further into

$$\mathbb{PT}^+ = \{\mathbf{Z} \in \mathbb{PT} \mid Z^\alpha \bar{Z}_\alpha > 0\}$$

$$\mathbb{PT}^- = \{\mathbf{Z} \in \mathbb{PT} \mid Z^\alpha \bar{Z}_\alpha < 0\}$$

with $\mathbb{PN}$ as common boundary.

Penrose (1967) has established various relationships between $\mathbb{PT}$ and $\mathbb{M}$ and its complexification $\mathbb{CM}$. Namely,

(A) A one-to-one relation between null lines in $\mathbb{M}$ and points in $\mathbb{PN}$.
(B) A one-to-one relation between points in $\mathbb{M}$ and projective lines in $\mathbb{PN}$.
(C) Therefore, two points in $\mathbb{M}$ have null separation if and only if their corresponding projective lines in $\mathbb{PN}$ intersect.
(D) Projective lines in $\mathbb{PT}$ that do not lie entirely in $\mathbb{PN}$ are in one-to-one correspondence with points in $\mathbb{CM}$.

Now, by using the above definitions we can establish the following.

> *Proposition* 2. A projective line $\mathbf{P} \subset \mathbb{PT}$ defines a *real* point of $\mathbb{CM}$ if and only if $\mathbf{P} = \mathscr{H}(\mathbf{P})$.

Proof. According to Definition 3, for a projective line

$$\mathbf{P} \subset \mathbb{PT}, \mathscr{H}: \mathbf{P} \mapsto \bar{\mathbf{P}}$$

Here

$$\begin{aligned}\bar{\mathbf{P}} \equiv \mathscr{H}(\mathbf{P}) = \{&\text{points } \mathbf{X} \in \mathbb{PT} \mid h(X^\alpha, R^\alpha) = 0,\ \forall R^\alpha = \lambda Y^\alpha + \mu Z^\alpha \\ &\text{where } \lambda, \mu \text{ are not both zero and twistors } Y^\alpha, Z^\alpha \\ &\text{define null lines meeting at the point } P \in \mathbb{M}\}\end{aligned}$$

By the dimension theorem for a correlation (Yale, 1968, p. 258) $\bar{\mathbf{P}}$ is a projective line.

Now, if $\mathbf{P} = \bar{\mathbf{P}}$ then every projective point $\mathbf{Z}$ lying on $\mathbf{P}$ also lies on its conjugate hyperplane $\bar{\mathbf{Z}}$, defined by $\mathscr{H}$. This follows from the *principle of duality* for projective spaces (Yale, 1968, p. 256), which implies that $\mathbf{Z}$ lies on $\mathbf{P}$ if and only if

$\bar{\mathbf{P}}$ lies on $\bar{\mathbf{Z}}$. Then, by Definition 4, $\mathbf{P} \subset \mathbb{CN}$ and therefore by (B) above defines a point $P \in \mathbb{M}$.

If P is a real point of $\mathbb{CM}$, then $P = \bar{P}$ and both light cones are identified. We can therefore state the equation following:

$$\{\text{points } \mathbf{Y} \in \mathbb{PT} \mid h(Y^\alpha, S^\alpha) = 0,\ \forall S^\alpha = \lambda V^\alpha + \mu U^\alpha, \text{ where } \lambda, \mu \text{ are not both zero and } V^\alpha, U^\alpha \text{ define null lines meeting at the point } \bar{P} \in \mathbb{M}\}$$
$$= \{\text{points } \mathbf{X} \in \mathbb{PT} \mid h(X^\alpha, R^\alpha) = 0,\ \forall R^\alpha = \lambda V^\alpha + \mu U^\alpha \cdots P \in \mathbb{M}\}$$

In other words,

$$\mathscr{H}(\bar{\mathbf{P}}) = \mathscr{H}(\mathbf{P})$$

or

$$\bar{\mathbf{P}} = \mathbf{P} \qquad \square$$

4. *Minkowski Space-Time and Lagrangian Subspaces of $T_{X^\alpha}\mathbb{T}$*

The results of Sections 2 and 3 contain two items of interest.

(i) Points of $\mathbb{M}$ are in one-to-one correspondence with projective lines $\mathbf{P} \subset \mathbb{PT}$ such that $\mathbf{P} = \mathscr{H}(\mathbf{P})$.

(ii) Lagrangian subspaces $F_{X^\alpha} \subset T_{X^\alpha}\mathbb{T}$ defined by polarizations F of $\mathbb{T}$ are interpreted projectively as projective lines $\mathbb{P}F_{X^\alpha} \subset \mathbb{P}T_{X^\alpha}\mathbb{T}$ such that $\mathbb{P}F_{X^\alpha} = \mathscr{C}(\mathbb{P}F_{X^\alpha})$.

We can expect that a map $I: \mathbb{P}T_{X^\alpha}\mathbb{T} \to \mathbb{PT}$ exists, taking a line $\mathbb{P}F_{X^\alpha}$ to a line $\mathbf{P} \subset \mathbb{PT}$ which passes through the projective point $\mathbf{X} \in \mathbb{PT}$. If such an induced line $\mathbf{P}$ has the property that $\mathbf{P} = \mathscr{H}(\mathbf{P})$ under the Hermitian correlation $\mathscr{H}$ on $\mathbb{PT}$, then it will define a point of $\mathbb{M}$. For this to be so, then $\mathbf{X}$ must be contained in $\mathbb{PN}$.

This prompts the conjecture that the assignment of a field of 2-dimensional isotropic subspaces $F_{X^\alpha} \subset T_{X^\alpha}\mathbb{T}$ to each $X^\alpha \in \mathbb{N} \subset \mathbb{T}$ determines a set of points in real Minkowski space-time.

Support for this conjecture can be obtained from physical considerations: Points in Minkowski space-time are uniquely specified by the intersection of null geodesic congruences with vanishing shear and rotation (the light cones). Equivalently, by rotation-free null congruences normal to families of 2-spheres (Newman and Winicour, 1974). Now one entity that induces a measure of the rotation of a null geodesic congruence in $\mathbb{M}$ is the symplectic structure on the cotangent bundle over $\mathbb{M}$ (Penrose, 1972, 1975). Crampin and Pirani (1971) have shown that this symplectic structure agrees with the symplectic structure on $\mathbb{T}$. We can therefore expect a Lagrangian subspace $F_{X^\alpha} \subset T_{X^\alpha}\mathbb{T}$ to define the set of momentum directions of a rotation-free null congruence in $\mathbb{M}$.

The vanishing shear property for a null congruence intersecting at a space-time point can be given a neat formulation in $\mathbb{PT}$ by means of the Kerr theorem (Penrose, 1967). Essentially, a shear-free null congruence is representable in $\mathbb{PT}$

by a submanifold $S \cap \mathbb{PN}$, where S is a complex analytic surface in $\mathbb{PT}$. It should be possible to determine the restrictions that this condition places on those projective lines in $\mathbb{PN}$ that are induced from the projective lines in $\mathbb{P}T_X{}^\alpha\mathbb{T}$ corresponding to the subspaces $F_X{}^\alpha \subset T_X{}^\alpha\,\mathbb{T}$.

Using the Hermitian correlation $\mathscr{H}$ on $\mathbb{PT}$ we can, by means of Definition 3, construct a *reciprocal* $\bar{S}$ to S, which is essentially the set of hyperplanes "conjugate" to the projective points of S. It turns out that the shear-free null congruences in $\mathbb{M}$ that are *also* rotation-free everywhere are those that are representable by *both* $S_\cap\mathbb{PN}$ and $\bar{S}_\cap\mathbb{PN}$.

The above conjecture can now be rephrased in a more restrictive form: An assignment of a field of Lagrangian subspaces $F_X{}^\alpha \subset T_X{}^\alpha\mathbb{T}$ to each $X^\alpha \in \mathbb{N} \subset \mathbb{T}$ determines a set of real space-time points. This should be so when their projective counterparts $\mathbb{P}F_X{}^\alpha$ correspond to lines $\mathbf{P} \subset \mathbb{PT}$ such that $\mathbf{P} \subseteq S_\cap$ *and* $\mathbf{P} \subseteq \bar{S}_\cap\mathbb{PN}$.

It should be noted that a complete understanding of the significance of polarizing $\mathbb{T}$ requires consideration of the involutive condition of Definition 1.

5. *Changing the Field of Lagrangian Subspaces on* $\mathbb{T}$

We discussed previously (Campbell and Dodson, 1976) the interpretation of nonsingular maps among Lagrangian subspaces of $\mathbb{R}^{2n}$ in terms of the action of the projective group $PGl(2n;\mathbb{R})$ on $\mathbb{R}P^{2n-1}$. Theorem 3 stated there will also be applicable to the complex case. So, nonsingular maps among Lagrangian subspaces $W \subset \mathbb{C}^{2n}$ are given by projective transformations in $\mathbb{C}P^{2n-1}$. Namely that those among the $(n-1)$-dimensional hyperplanes $\mathbb{P}W \subset \mathbb{C}P^{2n-1}$ that have the property that $\mathbb{P}W \subseteq \mathscr{C}(\mathbb{P}W)$.

In particular, for a polarization F of $\mathbb{T}$ we are interested in a projective interpretation for maps among Lagrangian subspaces $F_X{}^\alpha \subset T_X{}^\alpha\mathbb{T}$. These will obviously be given by the action of $PGl(4;\mathbb{C})$ on $\mathbb{P}T_X{}^\alpha\mathbb{T}$. More specifically, by projective transformations among those projective lines $\mathbb{P}F_X{}^\alpha \subset \mathbb{P}T_X{}^\alpha\mathbb{T}$ with the property $\mathbb{P}F_X{}^\alpha = \mathscr{C}(\mathbb{P}F_X{}^\alpha)$.

Consider a field of Lagrangian subspaces defined on $\mathbb{T}$. By the conjecture in Section 4 we can now hope for a suitable interpretation of maps among such fields in terms of sets of transformations on $\mathbb{M}$. Indeed, it should be possible to determine the space-time transformations involved in each case by examining the induced action on $\mathbb{PT}$ of the projective transformations among the $\mathbb{P}F_X{}^\alpha$. In fact, Penrose (1967) has discussed the correspondence between projective transformations and correlations of $\mathbb{PT}$ and transformations of $\mathbb{M}$, $\mathbb{CM}$, respectively. This can be summarized as follows:

(i) Projective transformations of $\mathbb{PT}$ that leave $\mathbb{PN}$ invariant correspond to the conformal transformations of $\mathbb{M}$ "continuous with the identity".

(ii) Projective transformations of $\mathbb{PT}$ combined with a complex conjugation operation correspond to space or time reflections on $\mathbb{M}$. For space reflections $\mathbb{PT}^-$ is interchanged with $\mathbb{PT}$ and for time reflections each is transformed into itself.

(iii) Correlations of $\mathbb{PT}$ correspond to transformations of $\mathbb{CM}$, and in fact certain combinations of correlations of $\mathbb{PT}$ can result in the transformations of (ii).

6. *Concluding Remarks*

This paper has been concerned with the use of projective geometry for the characterization of Lagrangian subspaces of $\mathbb{C}^{2n}$, maps among them being given by the projective group $PGl(2n;\mathbb{C})$.

In particular, we have discussed the possible consequences of this approach for twistor space which is $\mathbb{C}^4$.

The previous work of Penrose (1967) on the intimate relationship existing between projective twistor space and compactified Minkowski space-time was summarized and explicit definitions given for the entities used. This prompted some conjectures on a space-time interpretation for fields on $\mathbb{T}$ of Lagrangian subspaces $F_X{}^\alpha \subset T_X{}^\alpha\mathbb{T}$ and nonsingular maps among them.

One final point of interest: Penrose (1972, 1975) has argued that the conformal curvature of space-time shows up classically in terms of symplectic automorphisms of twistor space that "shift" its complex structure. This results in a nonzero shear for the null geodesics of the light cones after passage through a region of curvature. However, the zero-rotation property remains invariant. Now it is known that the symplectic automorphisms that can arise on a symplectic manifold (for example by the induced action of a change of polarization) have a projective counterpart, in fact, as projective transformations commuting with a symplectic correlation.

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